Math 601 Midterm 2 Solution

Please notify me if you see any typos/errors.

Question	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	100	

Question	Bonus Points	Score
Bonus Question 1	5	
Total:	5	

Question 1. (15 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) An $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.

(b) A square matrix P is orthogonal if and only if the columns of P form an orthonormal set.

(c) A square matrix is invertible if and only if its determinant is nonzero.

(d) A set of nonzero orthogonal vectors are always linearly independent.

(e) If $\Delta(t) = (t-2)^3(t+1)(t-4)$ is the characteristic polynomial of a matrix A, then A has at least 3 linearly independent eigenvectors.

Solution:

(a) False (e.g. the identity matrix)

(b) True

(c) True

(d) True

(e) True

Question 2. (10 pts)

Determine whether the matrix $A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ is diagonalizable.

Solution: Use cofactor expansion along the first column

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^3$$

So $\lambda = 2$ is an eigenvalue with multiplicity 3.

Now solve for the eigenvectors belonging to $\lambda = 2$, i.e. the kernel of the matrix

$$\begin{vmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix}$$

So there is only one linearly independent eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We need 3 linearly independent eigenvectors for A to be diagonalizable. By the above calculation, we see that A is not diagonalizable.

Question 3. (10 pts)

The eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ are $\lambda_1 = -2$ with $v_1 = (1, 1, 0)^T$, $\lambda_2 = -2$ with $v_2 = (1, 0, -1)^T$ and $\lambda_3 = 4$ with $v_3 = (1, 1, 2)^T$. Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

In other words,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-2t} + c_3 e^{4t} \\ c_1 e^{-2t} + c_3 e^{4t} \\ -c_2 e^{-2t} + 2c_3 e^{4t} \end{bmatrix}$$

Question 4. (15 pts)

Recall that $S = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2(t)$. Let $F : \mathbb{P}_2(t) \to \mathbb{P}_2(t)$ be the linear transformation defined by

$$F(1) = 1 + t^2, F(t) = 2 + t + t^2 \text{ and } F(t^2) = -1 + t - 2t^2$$

(a) Write down the matrix representation of F relative to the basis $S = \{1, t, t^2\}$.

Solution:

$$[F]_S = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(b) Find the kernel of F.

Solution: First reduce the matrix $[F]_S$ in part (a) to its echelon form, which is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\text{Ker}F = \text{span}\{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\}$. In other words, $\text{Ker}F$ is spanned of one polynomial $3 - t + t^2$.

(c) Find the dimension of the image of F.

Solution:

$$\dim(\text{Im}F) + \dim(\text{Ker}F) = 3$$

From part (b), we know that $\dim(\text{Ker}F) = 1$. So $\dim(\text{Im}F) = 2$.

(d) Is F is an isomorphism? Explain.

Solution: Since $\text{Ker}F$ is not equal to the zero vector space $\{0\}$, we see that F is not an isomorphism.

Question 5. (10 pts)

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 2-i \\ 2+i & 0 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2-i \\ 2+i & -\lambda \end{vmatrix} = -\lambda(4-\lambda) - 5 = (\lambda - 5)(\lambda + 1)$$

When $\lambda = 5$, the eigenvector is

$$v = \begin{bmatrix} (2-i) \\ 1 \end{bmatrix}$$

When $\lambda = -1$, the eigenvector is

$$w = \begin{bmatrix} -1 \\ (2+i) \end{bmatrix}$$

Question 6. (10 pts)

Let U be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (1, 7, 1, 7), v_2 = (0, 7, 2, 7)$ and $v_3 = (1, 8, 1, 6)$. Find an orthogonal basis of U.

Solution:

$$w_1 = v_1 = (1, 7, 1, 7)$$

$$w_2 = v_2 - \frac{\langle w_1, w_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = (-1, 0, 1, 0)$$

$$w_3 = v_3 - \frac{\langle w_1, w_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, w_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = (0, 1, 0, 1)$$

Question 7. (10 pts)

Let V be the vector space spanned by the basis $S = \{1, \cos t, \sin t\}$. Determine whether the functions

$$f_1(t) = 1 + 2\cos t + 3\sin t$$
$$f_2(t) = 2 + 5\cos t + 7\sin t$$
$$f_3(t) = 1 + 3\cos t + 5\sin t$$

are linearly independent or not. **(Hint: try to use the corresponding coordinate vectors with respect to the basis S.)**

Solution:

$$[f_1]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[f_2]_S = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

$$[f_3]_s = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{bmatrix}$$

its echelon form is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank 3. Therefore f_1, f_2 and f_3 are linearly independent.

Question 8. (10 pts)

Let $\mathbb{P}_2(t)$ be the vector space of polynomials with degree ≤ 2. Suppose the linear transformaiton $B : \mathbb{P}_2(t) \to \mathbb{P}_2(t)$ is defined by

$$B(p) = p(0) + p(1)t + p(2)t^2$$

for every polynomial $p \in \mathbb{P}_2(t)$. Note that here $p(0)$ (resp. $p(1), p(2)$) is the value of the polynomial $p(t)$ at $t = 0$ (resp. $t = 1, 2$). In particular, $p(0), p(1)$ and $p(2)$ are real numbers, and $p(0) + p(1)t + p(2)t^2$ is a polynomial in $\mathbb{P}_2(t)$. Show that B is an isomorphism. **(Hint: try to use the matrix repsentation of B relative to a basis.)**

Solution: Note that

$$B(1) = 1 + t + t^2$$
$$B(t) = t + 2t^2$$
$$B(t^2) = t + 4t^2$$

So the matrix representation of B relative to the basis $S = \{1, t, t^2\}$ is

$$[B]_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

its echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

which is invertible. So B is an isomorphism.

Question 9. (10 pts)

A function f on the complex plane $\mathbb{C}$ is defined by

$$f(z) = x^2 + x + 2ixy + iy - y^2,$$

where $z = x + iy$. Determine whether f is entire (that is, analytic on the whole complex plane $\mathbb{C}$).

Solution:

$$u(x, y) = x^2 + x - y^2$$
$$v(x, y) = 2xy + y$$

We have

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

Clearly, all $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on $\mathbb{C}$. Moreover, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied. Therefore, f is entire.

Bonus Question 1. (5 pts)
Consider the system of differential equations in Question 3 again:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. In Question 3, you have find the general solution $\mathbf{x}(t)$ of the system. Fidn find a specific solution $\mathbf{x}(t)$ such that

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

when $t = 0$. (Such a solution is called a solution of the above system with the given initial condition).

Solution: From the solution of Question 3, we have

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + c_3 \\ -c_2 + 2c_3 \end{bmatrix}.$$

Therefore, we need to solve for c_1, c_2 and c_3 of the following linear system

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + c_3 \\ -c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

which has a unique solution $c_1 = 2, c_2 = -3$ and $c_3 = -1$. So the solution satisfying the given initial condition is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -e^{-2t} - e^{4t} \\ 2e^{-2t} - e^{4t} \\ 3e^{-2t} - 2e^{4t} \end{bmatrix}$$